SEMISTABILITY AND HILBERT-KUNZ MULTIPLICITIES FOR CURVES

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1. Introduction

Let $(R, \mathbf{m})$ be a Noetherian local ring of dimension d and of prime characteristic $p > 0$, and let I be an $\mathbf{m}$-primary ideal. Then one defines the *Hilbert-Kunz function* of R with respect to I as

$$HK_{R,I}(p^n) = \ell(R/I^{(p^n)}),$$

where

$$\begin{aligned} I^{(p^n)} &= n\text{-th Frobenius power of } I \\ &= \text{ideal generated by } p^n\text{-th powers of elements of } I. \end{aligned}$$

The associated *Hilbert-Kunz multiplicity* is defined to be

$$HKM(R, I) = \lim_{n\to\infty} \frac{HK_{R,I}(p^n)}{p^{nd}}.$$

Similarly, for a non local ring R (of prime characteristic p), and an ideal $I \subseteq R$ for which $\ell(R/I)$ is finite, the Hilbert-Kunz function and multiplicity make sense. Henceforth for such a pair (R, I), we denote the Hilbert-Kunz multiplicity of R with respect to I by $HKM(R, I)$, or by $HKM(R)$ if I happens to be an obvious maximal ideal.

Given a pair $(X, \mathcal{L})$, where X is a projective curve over an algebraically closed field k of positive characteristic p, and $\mathcal{L}$ is a base point free line bundle $\mathcal{L}$ on X, define

$$HKM(X, \mathcal{L}) = \text{HK multiplicity of the section ring } B \text{ with respect to the ideal } B_1B,$$

where $B = \oplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n})$ and $B_1 = H^0(X, \mathcal{L})$. Note that when $\mathcal{L}$ is very ample, giving an embedding $X \longrightarrow \mathbf{P}^r_k$, then $HKM(X, \mathcal{L})$ equals the HK multiplicity of the "homogeneous coordinate ring" $A = \oplus A_n$, with respect to its maximal ideal $\oplus A_{n>0}$, where A is the image of the natural map ϕ, induced by $\mathcal{L}$,

$$\oplus_{n\geq 0} H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(n)) \xrightarrow{\phi} \oplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n}).$$

To discuss HK multiplicity of singular curves, we need to also consider the HK multiplicity of B with respect to the ideal generated by $W \subseteq H^0(X, \mathcal{L})$, where W is a base point free linear system, which we denote by

$$HKM(X, \mathcal{L}, W) = \text{HK multiplicity of } B \text{ with respect to the ideal generated by } W.$$

Notation 1.1. Now given $(X, \mathcal{L}, W)$ as above, where X is a nonsingular projective curve over k, consider the following short exact sequence

$$0 \longrightarrow V_{\mathcal{L}}(W) \longrightarrow W \otimes \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow 0, \tag{1.1}$$

where $V_{\mathcal{L}}(W)$ is a vector bundle of rank r = vector-space dimension of $W - 1$ and is the kernel of the surjective map $W \otimes \mathcal{O}_X \longrightarrow \mathcal{L}$. If $W = H^0(X, \mathcal{L})$ then we denote $V_{\mathcal{L}}(W)$ by $V_{\mathcal{L}}$.

In Section 2, we prove (see Proposition 2.5 and Remark 2.6) that if $V_{\mathcal{L}}$ is strongly semistable (i.e., the pull back of $V_{\mathcal{L}}$ under every iterated Frobenius map is semistable) then

$HKM(X, \mathcal{L}) =$ the HK multiplicity of the section ring with respect to its graded maximal ideal,

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(which may not be true in general without the strong semistability condition). We also give a lower bound for $HKM(X, \mathcal{L}, W)$ in terms of deg $\mathcal{L}$ and dim W, which is achieved when $V_{\mathcal{L}}(W)$ is strongly semistable. Later (see Theorem 4.14) we prove the converse of this.

One consequence of Proposition 2.5 is that for given $(X, \mathcal{L})$, if $HKM(X, \mathcal{L})$ does not achieve the lower bound, then $V_{\mathcal{L}}$ is not strongly semistable. For a plane curve X and $\mathcal{L} = \mathcal{O}_X(1)$, if X is nonsingular or singular with certain conditions on singularities then the referee provided a proof (Proposition 3.4, corollaries 3.5 and 3.6) that $V_{\mathcal{L}}$ is semistable.

In Section 4, which has been rewritten as per the suggestions of the referee, we prove that, for an arbitrary base-point free ample line bundle $\mathcal{L}$ on a nonsingular curve X of genus g (hence for any irreducible projective curve C), there is an expression for $HKM(X, \mathcal{L}, W)$ (for $HKM(C, \mathcal{O}_C(1))$) in terms of the ranks and degrees of the vector bundles occuring in a "strongly stable Harder-Narasimhan filtration" (in the sense of recent work of A. Langer [L]) of some Frobenius pullback of $V_{\mathcal{L}}(W)$ (see Theorem 4.12). Though this seems difficult to use in actually computing the HK multiplicity, except when $V_{\mathcal{L}}(W)$ is strongly semistable, it does imply that it is a rational number, for instance. We also prove the converse to the Section 2 result mentioned above.

In Section 5, we discuss plane curves. In general, Theorem 5.3 gives a formula (and hence bounds) for the HK multiplicity of an arbitrary plane curve C of degree d over a field of characteristic $p > 0$. In particular (Corollary 5.4) if X is a nonsingular plane curve of degree d then

$$HKM(X, \mathcal{O}_X(1)) = \frac{3d}{4} + \frac{l^2}{4dp^{2s}}$$

where $0 \leq l \leq d(d-3)$, and l is an integer congruent to pd (mod 2), and $s \geq 1$ (we allow $s = \infty$) is such that $F^{(s-1)*}V_{\mathcal{O}_X(1)}$ is semistable and $F^{s*}V_{\mathcal{O}_X(1)}$ is not semistable (here $s = \infty$ means that $V_{\mathcal{O}_X(1)}$ is strongly semistable,).

The formulas (for singular and nonsingular plane curves) also imply that for $p >> 0$ (for example when $p > d(d-3)$), one can recover the numbers s and l, where l is the measure of how much $F^{s*}V_{\mathcal{O}_X(1)}$ is destablized, in the sense that if $\mathcal{L}_1 \subset F^{s*}V_{\mathcal{O}_X(1)}$ is the Harder-Narasimhan filtration then slope $\mathcal{L}_1 =$ slope $F^{s*}V_{\mathcal{O}_X(1)} + l/2$. So in this case, we have a simple numerical characterization of semistablity of the kernel bundle under the Frobenius map via HK multiplicity.

Using this, and Monsky's results ([M1], [M3]), which are explicit computations for certain nonsingular quartics), we prove the following (see Proposition 5.10): for any integer $n \geq 1$, there exist explicit rank 2 vector bundles V on nonsingular curves of genus 3 over a field of characterstic 2 or 3, such that $F^{n*}V$ is semistable, but $F^{(n+1)*}V$ is not semistable. Morevover, when $p = 3$, the result also holds for $n = 0$.

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Some of our results, particularly the formula for HK multiplicity in Theorem 4.12, are also contained in an equivalent form in a recent preprint of H. Brenner [B]. Our results here have been obtained concurrently, and independently. The rationality of the HK multiplicity of a smooth plane curve had been also proved by Monsky (unpublished), by different methods (private communications).

2. Semistability and HK multiplicity

We first recall the notion of semistability. If V is a vector bundle of rank r on a projective curve X, recall that deg $V :=$ deg $(\wedge^r V)$, and slope $(V) := \mu(V) =$ deg $V/$rank V.

Definition 2.1. Let V be a vector bundle of rank r on a projective curve X. Then V is *semistable* if for any subbundle $V' \hookrightarrow V$, we have
$$\mu(V') \leq \mu(V).$$

Definition 2.2. A vector bundle V on X is called *strongly semistable* if $F^{s*}V$ is semistable for the s^{th} iterate of the absolute Frobenius map, $F^s : X \longrightarrow X$, for all $s \geq 0$.

Remark 2.3. If W is a line bundle then it is semistable, and if V is a semistable bundle then so are $V^\vee$ and $V \otimes W$.

From now onwards, X is a nonsingular (projective) curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$ and $\mathcal{L}$ is a base point free line bundle on X, unless stated otherwise. Recall the notation $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$, for any coherent sheaf $\mathcal{F}$ on X, and $i = 0, 1$.

Lemma 2.4. *Let X be a nonsingular projective curve of genus g and V be a semistable bundle on X of rank r and degree d. Then*
1. *If* $\deg\ W < 0$ *then* $h^0(X, W) = 0$,
2. *If* $\deg\ W > r(2g-2)$ *then* $h^1(X, W) = 0$ *and* $h^0(X, W) = \deg\ W - r(g-1)$.
3. *If* $0 \leq \deg\ W \leq r(2g-2)$ *then* $h^0(X, W) \leq rg$,

Proof. Statement (1) follows from the definition of semistable vector bundle.

By Serre duality, we have $h^1(X, W) = h^0(X, \omega_X \otimes W^\vee)$. Since $\omega_X \otimes W^\vee$ is semistable, we get $h^0(X, \omega_X \otimes W^\vee) = 0$ if $\deg\ W > r(2g-2)$, hence $h^1(X, W) = 0$. This, and the Riemann-Roch formula
$$h^0(X, W) - h^1(X, W) = \deg\ W + r(1-g),$$
implies statement (2).

To prove statement (3), we choose a line bundle $\mathcal{L}$, given by an effective divisor of degree 1, and an integer $m \geq 0$ such that $\deg\ (W \otimes \mathcal{L}^m) \leq r(2g-2)$ and $\deg\ (W \otimes \mathcal{L}^{m+1}) > r(2g-2)$. Now
$$\begin{aligned} h^0(X, W) \leq h^0(X, W \otimes \mathcal{L}^{m+1}) &= h^1(X, W \otimes \mathcal{L}^{m+1}) + \deg\ (W \otimes \mathcal{L}^{m+1}) + r(1-g) \\ &= \deg\ (W \otimes \mathcal{L}^m) + r + r(1-g) \leq rg. \end{aligned}$$
This proves statement (3). $\square$

Proposition 2.5. *Let X be a nonsingular projective curve of genus g and let $\mathcal{L}$ be a base point free line bundle of degree d on X. If $V_{\mathcal{L}}$ (see (1.1)) is strongly semistable then*
$$HKM(X, \mathcal{L}) = HKM(B, \mathbf{m}) = \frac{dh}{2(h-1)},$$
where $h = h^0(X, \mathcal{L})$, $B = \oplus_{n \geq 0} H^0(X, \mathcal{L}^n)$ and $\mathbf{m} = \oplus_{n>0} H^0(X, \mathcal{L}^n)$ is the graded maximal ideal of B.

Proof. Let $B_n = H^0(X, \mathcal{L}^n)$. Consider the Frobenius twisted multiplication map,
$$\mu_{k,n} : B_k^{(q)} \otimes B_{n-kq} \longrightarrow B_n$$
given by $r \otimes r' \to r^q r'$, where $r \in B_k$ and $r' \in B_{n-kq}$ and $B_k^{(q)} = B_k$ as an additive group with k-action on it given by $\lambda \cdot r = \lambda^q r$ for $\lambda \in k$ and $r \in B_k$. Now
$$\ell(B/\mathbf{m}^{(q)}) = \sum_n \ell(B_n / \sum_k \text{im}\ \mu_{k,n}).$$
Consider the short exact sequence
$$0 \longrightarrow V_{\mathcal{L}} \longrightarrow H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow 0,$$
This gives
$$0 \longrightarrow F^{s*}V_{\mathcal{L}} \otimes \mathcal{L}^{\otimes n} \longrightarrow H^0(X, \mathcal{L})^{(q)} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes n+q} \longrightarrow 0,$$
where $q = p^s$ and $F : X \longrightarrow X$ is the Frobenius map.

Hence we have a long exact sequence of cohomologies

$$H^0(X, F^{s*}V_{\mathcal{L}}\otimes\mathcal{L}^{\otimes n}) \longrightarrow H^0(X,\mathcal{L})^{(q)}\otimes H^0(X,\mathcal{L}^{\otimes n}) \longrightarrow H^0(X,\mathcal{L}^{\otimes n+q}) \longrightarrow H^1(X, F^{s*}V_{\mathcal{L}}\otimes\mathcal{L}^{\otimes n}),$$

where the second arrow is given by the map $\mu_{1,n+q}$.

Now rank $V_{\mathcal{L}} = h-1$, and

$$\begin{array}{rcl} \deg\,(F^{s*}V_{\mathcal{L}}\otimes\mathcal{L}^n) & = & \deg\,(F^{s*}V_{\mathcal{L}}) + (h-1)\deg\mathcal{L}^n \\ & = & q\deg\; V_{\mathcal{L}} + (h-1)n(d) \\ & = & (-q+(h-1)n)d \end{array}$$

Case 1 Suppose $n < q/(h-1)$. Then $\deg\,(F^{s*}V_{\mathcal{L}}\otimes\mathcal{L}^n) < 0$. Hence by Lemma 2.4, the map $\mu_{1,n+q}$ is injective.

Moreover $n+q-kq < q/(h-1)+q-kq \leq 0$, if $k\geq 2$. In particular im $\mu_{k,n+q} = 0$ for $k\geq 2$. Hence in this range $\ell(B_{n+q}/\sum_k \text{im}\,(\mu_{k,n+q})) = \ell(B_{n+q}/\text{im}\,(\mu_{1,n+q})) = \ell(B_{n+q}) - \ell(B_1)\cdot\ell(B_n)$.

Case 2 Suppose $n > q/(h-1)+(2g-2)/d$. Then $\deg\,(F^{s*}V_{\mathcal{L}}\otimes\mathcal{L}^n) > (h-1)(2g-2)$, hence by Lemma 2.4, the map $\mu_{1,q}$ is surjective, which implies $\ell(B_{n+q}/\text{im}\,(\mu_{1,n+q})) = 0$. Hence $\ell(B_{n+q}/\sum_k \text{im}\,(\mu_{k,n+q})) = 0$.

Case 3 Suppose $q/(h-1) \leq n \leq q/(h-1)+(2g-2)/d$. Then

$$0 \leq \deg\,(F^{s*}V_{\mathcal{L}}\otimes\mathcal{L}^n) \leq (h-1)(2g-2),$$

and therefore

$$\sum_{n=\lfloor q/(h-1)\rfloor}^{\lfloor q/(h-1)+(2g-2)/d\rfloor} h^0(X, F^{s*}V_{\mathcal{L}}\otimes\mathcal{L}^n) \leq (h-1)g\left(\frac{2g-2}{d}+1\right).$$

Therefore we have

$$HKM(X,\mathcal{L}) = HKM(B,\mathbf{m}) = \lim_{q\to\infty}\frac{1}{q^2}\sum_{n\geq 0}\ell(\frac{B_n}{\text{im}(\mu_{1,n})}) = \lim_{q\to\infty}\frac{1}{q^2}\sum_{n\geq -q}\ell(\frac{B_n}{\text{im}(\mu_{1,n+q})})$$

$$= \lim_{q\to\infty}\frac{1}{q^2}\sum_{-q\leq n}\left(h^0(X,\mathcal{L}^{n+q}) - h^0(X,\mathcal{L})h^0(X,\mathcal{L}^n) + h^0(X, F^{s*}V_{\mathcal{L}}\otimes\mathcal{L}^n)\right)$$

$$= \lim_{q\to\infty}\frac{1}{q^2}\sum_{-q\leq n\leq q/(h-1)} h^0(X,\mathcal{L}^{n+q}) - h^0(X,\mathcal{L})h^0(X,\mathcal{L}^n)$$

$$'' = \lim_{q\to\infty}\frac{1}{q^2}\sum_{0\leq n\leq q/(h-1)+q}\chi(X,\mathcal{L}^n) - h\sum_{0\leq n\leq q/(h-1)}\chi(X,\mathcal{L}^n) = (dh)/2(h-1)$$

This proves the proposition. □

Remark 2.6. In the above proof, replacing the complete linear system by any base point free linear system W of $\mathcal{L}$, of vector-space dimension $r+1$ (and replacing h by $r+1$ everywhere), one sees that if $V_{\mathcal{L}}(W)$ is strongly semistable then $HKM(X,\mathcal{L},W) = d(r+1)/2r$.

3. Applications and examples

In this section X is a nonsingular curve and $\mathcal{L}$ is a base point free line bundle on X, and $V_{\mathcal{L}}$ is the kernel vector bundle given by the natural map

$$0\longrightarrow V_{\mathcal{L}}\longrightarrow H^0(X,\mathcal{L})\otimes\mathcal{O}_X\longrightarrow\mathcal{L}\longrightarrow 0.$$

We use the following notation in this and in the forthcoming sections.

Notation 3.1. C denotes an irreducible curve of degree $d > 1$, over an algebraically closed field of characteristic p and $\pi : X_C \longrightarrow C$ is the normalization of C, where g is the genus of X_C and $\mathcal{L}_C = \pi^*\mathcal{O}_C(1)$ and $W_C = H^0(C, \mathcal{O}_C(1))$. Note that $W_C \subset H^0(X_C, \mathcal{L}_C)$ is a base point free linear system. Hence this gives a natural short exact sequence of $\mathcal{O}_{X_C}$-modules

$$0 \longrightarrow V_C \longrightarrow W_C \otimes \mathcal{O}_{X_C} \longrightarrow \mathcal{L}_C \longrightarrow 0, \tag{3.1}$$

where $V_C = V_{\mathcal{L}_C}(W_C)$ following our earlier Notation 1.1.

Remark 3.2. Since π is a finite birational map, by lemma 1.3 in [M0], theorem 2.7 in [WY] or in [BCP], we have

$$HKM(C, \mathcal{O}_C(1)) = HKM(X_C, \mathcal{L}_C, W_C).$$

Here we discuss some examples $(X, \mathcal{L})$ for which the vector bundle $V_{\mathcal{L}}$ is strongly semistable. But before that we need to check the first necessary condition, i.e., that the vector bundle $V_{\mathcal{L}}$ is itself semistable. The referee has provided the proofs of Proposition 3.4 and its Corollaries 3.5 and 3.6. Before coming to that we recall the following definition.

Definition 3.3. The *gonality* of a nonsingular curve X is the least integer d, for which there exists a line bundle of degree d with a base point free complete linear system of projective dimension 1 (in other words a line bundle of degree d which induces a nonconstant map $X \to \mathbf{P}^1$).

Proposition 3.4. *If X_C has gonality $\geq d/2$ then $V_{\mathcal{L}}$ is semistable.*

Proof. If $V_{\mathcal{L}}$ is not semistable, then neither is $V_{\mathcal{L}}^\vee$. Hence there exists a quotient line bundle $\mathcal{L}_1$ of $V_{\mathcal{L}}^\vee$ such that $\mu(\mathcal{L}_1) < \mu(V_{\mathcal{L}}^\vee) = d/2$. Since $V_{\mathcal{L}}^\vee$ is globally generated, the line bundle $\mathcal{L}_1$ is globally generated. Now $\mathcal{L}_1$ cannot be the trivial bundle; otherwise we will have $\mathcal{O}_X \hookrightarrow V_{\mathcal{L}}$ which would imply that $H^0(X, V_{\mathcal{L}}) \neq 0$. So $h^0(X, \mathcal{L}_1) \geq 2$. So it follows that X has a line bundle, of degree $< d/2$, with a linear system of vector-space dimension ≥ 2, hence a line bundle of degree $< d/2$ with a base point free complete linear system of vector space dimension 2. In other words the gonality of $X < d/2$, which contradicts the hypothesis. This proves the proposition. $\square$

Corollary 3.5. *If X is a nonsingular plane curve, then $V_{\mathcal{L}}$, where $\mathcal{L} = \mathcal{O}_X(1)$, is semistable.*

Proof. A classical result of M. Noether (see [H] , theorem 2.1) implies that the gonality of X is $d-1$, where d is the degree of X. Now the proof follows from Proposition 3.4. $\square$

Corollary 3.6. *Suppose C is an irreducible projective plane curve of degree d such that the only singularities of C are nodes and cusps, that $d \geq 4$ and the number of singularities, δ, satisfies $1 \leq \delta \leq d-2$. Then V_C is semistable.*

Proof. Theorem 2.1 of [CK] implies (for $k = 1$ in their notation) that the gonality of X_C is $\geq d-2$. Hence once again the proof follows from Proposition 3.4. $\square$

In this context, we would also like to recall the following result given in [T1], which was the main ingredient in proving a conjecture of Monsky (see Remark 5.6 of this paper).

Proposition 3.7. *Let C be an irreducible projective plane curve of degree d with a singularity of multiplicity $r \geq d/2$. Then:*

(1) *if $r = d/2$ then V_C is strongly semistable,*
(2) *if $r > d/2$ then V_C is not semistable and its destabilizing line bundle is of degree $= r-d$.*

4. HK multiplicities for base point free line bundles

In this section, we consider $HKM(X, \mathcal{L}, W)$ where X is any non-singular projective curve of genus g over an algebraically closed field k of characteristic $p > 0$, and $\mathcal{L}$ is a line bundle on X of degree d with base point free linear system W. We derive an expression for the HK multiplicity in this case, involving terms which seem to be very difficult to compute, but which

shows that it is a rational number, with a denominator of a particular form. As a consequence (see Remark 3.2) the rationality of the HK multiplicity of an irreducible projective curve follows.

As mentioned in the introduction, this result was obtained independently by H. Brenner [B]. The tools, both in Brenner's proof and ours, are Lemma 2.4, Lemma 4.10, and a recent result of A. Langer [L] (Theorem 4.5). We shall also give a converse to our Remark 2.6.

Definition 4.1. Given a vector bundle E on X, a filtration by vector subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_t \subset E_{t+1} = E$$

is called a *Harder-Narasimhan filtration* (HN filtration) if

(i) $E_1, E_2/E_1, \ldots, E_{t+1}/E_t$ are semistable vector bundles,
(ii) $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_{t+1}/E_t)$.

Remark 4.2. Note that such a filtration exists and is unique (see [HN], lemma 1.3.7). Moreover, if $t \geq 1$, then

$$\mu(E_i) > \mu(E_i/E_{i-1}), \text{ for all } 2 \leq i \leq t+1.$$

The case when E is semistable corresponds to $t = 0$.

Notation 4.3. If $0 \subset E_1 \subset \cdots \subset E_t \subset E_{t+1} = E$ is the HN filtration of E then we write

$$\mu_{\max}(E) = \mu(E_1) \text{ and } \mu_{\min}(E) = \mu(E/E_t).$$

Definition 4.4. A filtration of subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_t \subset E_{t+1} = E$$

of E is a *strongly stable HN filtration* if it is a HN filtration and $E_1, E_2/E_1, \ldots, E_{t+1}/E_t$ are strongly semistable vector bundles.

Note that whenever E has a strongly stable HN filtration then the HN-filtration of $F^{k*}(E)$ is

$$0 \subset F^{k*}(E_1) \subset F^{k*}(E_2) \subset \cdots \subset F^{k*}(E_t) \subset F^{k*}(E_{t+1}) = F^{k*}(E).$$

Now recall the crucial result of Langer [L], which we state for the special case of curves.

Theorem 4.5. *(A. Langer) If V is a vector bundle on a nonsingular projective curve defined over an algebraically closed field of characteristic $p > 0$, then there exist $s > 0$ such that $F^{s*}(V)$ has a strongly stable HN filtration.*

Definition 4.6. For a vector bundle V on X, and an ample line bundle $\mathcal{L}$ on X, we define

$$\sigma_s(V) = \sum_{n \leq 0} h^0(F^{s*}(V) \otimes \mathcal{L}^n) + \sum_{n > 0} h^1(F^{s*}(V) \otimes \mathcal{L}^n).$$

Lemma 4.7. *If V is a strongly semistable vector bundle of rank r and degree a, and* $\deg\ \mathcal{L} = d$, *then*

$$\sigma_s(V) = \frac{a^2}{2rd} p^{2s} + O(p^s).$$

Proof. Suppose for example that $a \geq 0$. We are given that $F^{s*}(V) \otimes \mathcal{L}^n$ is semistable of degree $p^s a + rdn$. We choose $s > 0$ such that $(2g-2)/d < p^s a/rd$. Then

$$\sigma_s(V) = \sum_{n < \frac{-p^s a}{rd}} h^0(X, F^{s*}(V) \otimes \mathcal{L}^n) + \sum_{\frac{-p^s a}{rd} \leq n \leq \frac{2g-2}{d} - \frac{p^s a}{rd}} h^0(X, F^{s*}(V) \otimes \mathcal{L}^n)$$

$$+ \sum_{\frac{2g-2}{d} - \frac{p^s a}{rd} < n \leq 0} h^0(X, F^{s*}(V) \otimes \mathcal{L}^n) + \sum_{n > 0} h^1(X, F^{s*}(V) \otimes \mathcal{L}^n).$$

Now applying Lemma 2.4 to this equation we get

$$\sigma_s(V) = C_0 + \sum_{\frac{2g-2}{d} - \frac{p^s a}{rd} < n \leq 0} h^0(X, F^{s*}(V) \otimes \mathcal{L}^n) = C_0 + \sum_{\frac{2g-2}{d} - \frac{p^s a}{rd} < n \leq 0} \chi(X, F^{s*}(V) \otimes \mathcal{L}^n),$$

where $0 \leq C_0 \leq rg((2g-2)/d+1)$. This gives $\sigma_s(V) = \frac{a^2}{2rd}p^{2s} + O(p^s)$. The argument for $a < 0$ is similar. $\square$

Notation 4.8. To generalize Lemma 4.7 to an arbitrary vector bundle V on X, we shall attach a rational number $\alpha(V)$ to V, as follows. We choose $m \geq 0$ such that the vector bundle $F^{m*}V$ has a strongly stable HN filtration (this is possible by Theorem 4.5),

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = F^{m*}V,$$

Recall that, for any $n \geq 0$,

$$0 \subset F^{n*}E_1 \subset F^{n*}E_2 \subset \cdots \subset F^{n*}E_t \subset F^{n*}E_{t+1} = F^{(m+n)*}V,$$

is the strongly stable HN filtration of $F^{(m+n)*}V$. We set

$$a_i = p^{-m}\mathrm{deg}(E_i/E_{i-1}),\ r_i = \mathrm{rank}(E_i/E_{i-1})$$

$$\alpha(V) = \sum_i (a_i^2/r_i). \tag{4.1}$$

Remark 4.9. Note that these numbers are independent of the choice of m, and that

$$\sum a_i = a, \text{ and } \sum r_i = r.$$

Lemma 4.10. *Let $0 \to U \to V \to W \to 0$ be an exact sequence of vector bundles on X. Suppose that U and V admit strongly stable HN filtrations, and that*

$$\mu_{\min}(U) - \mu_{\max}(W) > \max(0, 2g-2).$$

Then $\sigma_s(V) = \sigma_s(U) + \sigma_s(W)$ for all s.

Proof. It suffices to show that

$$h^0(X, F^{s*}(V) \otimes \mathcal{L}^n) = h^0(X, F^{s*}(U) \otimes \mathcal{L}^n) + h^0(X, F^{s*}(W) \otimes \mathcal{L}^n)$$

for all s and n. Consider the canonical long exact sequence

$$0 \longrightarrow H^0(F^{s*}(U)\otimes\mathcal{L}^n) \longrightarrow H^0(F^{s*}(V)\otimes\mathcal{L}^n) \longrightarrow H^0(F^{s*}(W)\otimes\mathcal{L}^n) \longrightarrow H^1(F^{s*}(U)\otimes\mathcal{L}^n) \longrightarrow .$$

Now

$$\mu_{\min}(F^{s*}(U) \otimes \mathcal{L}^n) - \mu_{\max}(F^{s*}(W) \otimes \mathcal{L}^n) = p^s(\mu_{\min}(U) - \mu_{\max}(W) > 2g-2.$$

Therefore, either $\mu_{\max}(F^{s*}(W) \otimes \mathcal{L}^n) < 0$, in which case $h^0(F^{s*}(W) \otimes \mathcal{L}^n) = 0$, or

$$\mu_{\min}(F^{s*}(U) \otimes \mathcal{L}^n) > 2g-2,$$

in which case, we have $h^1(F^{s*}(U) \otimes \mathcal{L}^n) = 0$, by Serre duality. Hence the lemma follows, by the above long exact sequence. $\square$

Corollary 4.11. *For any vector-bundle V on X,*

$$\sigma_s(V) = \frac{\alpha(V)}{2d}p^{2s} + O(p^s).$$

Proof. Taking large enough Frobenius pull backs, *i.e.* for $m >> 0$, we can make sure that

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = F^{m*}V$$

is the strongly stable HN filtration of $F^{m*}V$ and

$$\mu(E_i/E_{i-1}) - \mu(E_{i+1}/E_i) > r(2g-2),$$

hence, by Remark 4.2,

$$\mu(E_i) - \mu(E_{i+1}/E_i) > r(2g-2).$$

Moreover, E_{i+1}/E_i is strongly semistable and $0 \subset E_1 \subset \cdots \subset E_i$ is the strongly stable HN filtration of E_i. Hence applying Lemma 4.10, for $s - m > 0$ we get

$$\sigma_{s-m}(E_{i+1}) = \sigma_{s-m}(E_i) + \sigma_{s-m}(E_{i+1}/E_i).$$

Now, for $s-m >> 0$, by induction

$$\sigma_s(V) = \sigma_{s-m}(E_{t+1}) = \sigma_{s-m}(E_1) + \sigma_{s-m}(E_2/E_1) + \cdots \sigma_{s-m}(E_{t+1}/E_t).$$

Now the corollary follows from Lemma 4.7. $\square$

Theorem 4.12. *Let $X \subset \mathbb{P}^r$ be a nonsingular projective curve over k and let $\mathcal{L}$ be a line bundle on X of degree d, with a base point free linear system W. Then*

$$HKM(X, \mathcal{L}, W) = (1/2d)(d^2 + \alpha(V_{\mathcal{L}}(W))).$$

In particular $HKM(X, \mathcal{L}, W)$ is a rational number.

Proof. Let B be the section ring $\oplus_{n\geq 0} H^0(X, \mathcal{L}^n)$, and I be the ideal of B generated by $W \cdot B$. We only need show that the HK multiplicity of B with respect to I is $(1/2d)(d^2 + \alpha(V_{\mathcal{L}}(W)))$. Making use of the various exact sequences

$$0 \to F^{s*}(V_{\mathcal{L}}(W)) \otimes \mathcal{L}^n \to \mathcal{L}^n \oplus \cdots \oplus \mathcal{L}^n \to \mathcal{L}^{n+p^s} \to 0,$$

one finds that

$$\dim \frac{B}{I^{[p^s]}B} = \sum_n (h^0(X, F^{s*}(V_{\mathcal{L}}(W)) \otimes \mathcal{L}^n) - (r+1)h^0(X, \mathcal{L}^n) + h^0(X, \mathcal{L}^{n+p^s})).$$

Now each term in this sum is unchanged when h^0 is replaced by h^1. So the sum is

$$\sigma_s(V_{\mathcal{L}}(W)) - (r+1)\sigma_s(\mathcal{O}_X) + \sigma_s(\mathcal{L}).$$

Since $\alpha(\mathcal{O}_X) = 0$ and $\alpha(\mathcal{L}) = d^2$, by Corollary 4.11, we have

$$\dim(B/I^{[p^s]}B) = \frac{1}{2d}(\alpha(V_{\mathcal{L}}(W)) + d^2)p^{2s} + O(p^s).$$

This proves the theorem. $\square$

Remark 4.13. We have

$$\frac{b^2}{s} + \frac{c^2}{t} - \frac{(b+c)^2}{s+t} = \frac{(cs-bt)^2}{st(s+t)}.$$

So if $s, t > 0$,

$$\frac{b^2}{s} + \frac{c^2}{t} \geq \frac{(b+c)^2}{s+t},$$

with equality if and only if $b/s = c/t$. It follows that $\alpha(V_{\mathcal{L}}(W)) \geq d^2/r$ with equality if and only if $V_{\mathcal{L}}(W)$ is strongly semistable. Together with Theorem 4.12, this gives:

Theorem 4.14. *For a nonsingular projective curve X with a line bundle $\mathcal{L}$ of degree d and a base point free linear system W, of $\mathcal{L}$, of dimension r,*

$$HKM(X, \mathcal{L}, W) \geq d(r+1)/2r,$$

and

$$HKM(X, \mathcal{L}, W) = d(r+1)/2r$$

if and only if $V_{\mathcal{L}}(W)$ is strongly semistable.

Now, Remark 3.2 implies the following

Corollary 4.15. *If $C \subseteq \mathbf{P}^r$ is an irreducible projective curve of degree d then*

$$HKM(C, \mathcal{O}_C(1)) = (1/2d)(d^2 + \alpha(V_C)),$$

which is a rational number. Furthermore

$$HKM(C, \mathcal{O}_C(1)) \geq d(r+1)/2r,$$

with equality if and only if V_C is strongly semistable

Corollary 4.16. *If X is a nonsingular projective curve of genus $g \geq 2$ and ω_X is the canonical sheaf of X then*

$$HKM(X, \omega_X) \geq g,$$

with equality if and only if V_{ω_X} is stongly semistable.

5. HK multiplicity for plane curves

In this section we use the Notation 3.1, where C is an irreducible plane curve of degree $d > 1$, over an algebraically closed field of characteristic p. Hence we have a natural short exact sequence of $\mathcal{O}_{X_C}$-modules

$$0 \longrightarrow V_C \longrightarrow W \otimes \mathcal{O}_{X_C} \longrightarrow \mathcal{L}_C \longrightarrow 0,$$

where $V_C = V_{\mathcal{L}}(W)$ is a rank two vector bundle.

Remark 5.1. For a rank two vector bundle V, either the bundle is strongly semistable or some iterated Frobenius pull back has HN filtration given by a line bundle $\mathcal{L} \subset F^{s*}V$ such that $F^{s*}V/\mathcal{L}$ is also a line bundle. In other words the HN filtration of $F^{s*}V$ is a strongly stable HN filtration. Hence the result of Langer is obvious.

The following lemma is proved in [SB], Corollary 2^p (see also [L]). We sketch another proof.

Lemma 5.2. *Let X be a nonsingular curve of genus g over an algebraically closed field k of characteristic $p > 0$. Let V be a vector bundle of rank 2 over X. Suppose there exists an exact sequence*

$$0 \to \mathcal{L}_1 \to F^*V \to \mathcal{M}_1 \to 0,$$

such that $\mathcal{L}_1$, $\mathcal{M}_1$ are line bundles, and

$$\deg\ \mathcal{L}_1 - \deg\ \mathcal{M}_1 > \max\ (2g-2, 0).$$

Then V is not semistable.

Proof. If $g = 0$ and V is semistable then $F^*(V)$ is semistable. This contradicts the hypothesis that $\deg\ \mathcal{L}_1 - \deg\ \mathcal{M}_1 > 0$. So we may assume that $g > 0$. Hence $\deg\ \mathcal{L}_1 - \deg\ \mathcal{M}_1 > 2g-2$. Then there is a canonical connection $\nabla : F^*(V) \longrightarrow F^*(V) \otimes \omega_X$ given locally by

$$\nabla(F^*(e_1)) = \nabla(F^*(e_2)) = 0,$$

where $\{e_1, e_2\}$ is any local basis for V. Let $f = p \circ \nabla\mid_{\mathcal{L}_1}$, where $p : F^*(V) \otimes \omega_X \longrightarrow \mathcal{M}_1 \otimes \omega_X$ is the obvious map. Let a and s be local sections of $\mathcal{O}_X$ and $\mathcal{L}_1$ respectively. Then

$$f(as) = p(s \otimes da + a\nabla s) = p(a\nabla s) = af(s).$$

Hence $f : \mathcal{L}_1 \longrightarrow \mathcal{M}_1 \otimes \omega_X$ is an $\mathcal{O}_X$-linear map.

If $f \neq 0$ then $\deg\ \mathcal{L}_1 \leq \deg\ \mathcal{M}_1 + (2g-2)$ which would contradict the hypothesis. Hence $f = 0$. Now, note that locally, $\mathcal{L}_1$ is a free $\mathcal{O}_X$-module of rank 1 in F^*V, generated by a section of the form $s = aF^*e_1 + F^*e_2$, or of the form $s = F^*e_1 + bF^*e_2$. Without loss of generality one can assume $s = aF^*e_1 + F^*e_2$. Then $f(s) = 0$ implies $F^*e_1 \otimes da \in \mathcal{L}_1 \otimes \omega_X$. Hence we can find a local section w of ω_X such that $F^*e_1 \otimes da = (aF^*e_1 + F^*e_2) \otimes w$, which implies $w = 0$ and $da = 0$. Hence $a = \widetilde{a}^p$ for some local section $\widetilde{a}$ of $\mathcal{O}_X$. This implies $aF^*e_1 + F^*e_2 = F^*(\widetilde{a}e_1 + e_2)$. Hence $\mathcal{L}_1 = F^*\mathcal{L}_1'$ for some line sub-bundle $\mathcal{L}_1'$ of V. Since $\deg\ F^*(\mathcal{L}_1') > 1/2 \deg\ F^*(V)$ we have $\deg\ \mathcal{L}_1' > \mu(W)$, which implies that V is not semistable. $\square$

Theorem 5.3. *Let C be an irreducible plane curve of degree $d > 1$. Let $X_C \xrightarrow{\pi} C$ be the normalization of C. Let V_C be the rank two vector bundle given by the natural map*

$$0 \longrightarrow V_C \longrightarrow H^0(C, \mathcal{O}_C(1)) \otimes \mathcal{O}_X \longrightarrow \mathcal{L}_C \longrightarrow 0.$$

Then one of the following holds:

1. *V_C is strongly semistable. In this case $HKM(C) = 3d/4$.*
2. *V_C is not semistable. Then*

$$HKM(C) = \frac{3d}{4} + \frac{l^2}{4d},$$

where $0 < l < d$ and l is an integer congruent to d (mod 2).

(3) *V_C is semistable but not strongly semistable. Let $s \geq 1$ be the number such that $F^{(s-1)*}V_C$ is semistable and $F^{s*}V_C$ is not semistable. Then*

$$HKM(C) = \frac{3d}{4} + \frac{l^2}{4dp^{2s}},$$

where l is an integer congruent to pd (mod 2) with $0 < l \leq 2g-2$, so that in particular $0 < l \leq d(d-3)$.

Proof. Case (1) follows from Remark 2.6 with $r = 2$.
Case (2) Given that V_C is not semistable, we have

$$0 \to \mathcal{L}_1 \to V_C \to \mathcal{M}_1 \to 0$$

where

$$\mu(\mathcal{L}_1) = \deg\ \mathcal{L}_1 = -\frac{d}{2} + \frac{l}{2} \text{ and } \mu(\mathcal{M}_1) = \deg\ \mathcal{M}_1 = -\frac{d}{2} - \frac{l}{2},$$

for some $l > 0$ and l is an integer congruent to $d(mod\ 2)$. Since this is the strongly stable HN filtration (see Remark 5.1), by Theorem 4.12

$$HKM(C) = \frac{3d}{4} + \frac{l^2}{4d}.$$

Since an irreducible plane curve of degree $d > 1$ has HK multiplicity $< d$, we have $l < d$. This proves the statement (2).
Case (3). If $\mathcal{L}_1$ is the destabilizing bundle of $F^{s*}V_C$ then there exists a short exact sequence

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow F^{s*}V_C \longrightarrow \mathcal{M}_1 \longrightarrow 0,$$

such that for some positive integer l

$$\deg\ \mathcal{M}_1 = -\frac{d}{2}p^s - l/2, \text{ and } \deg\ \mathcal{L}_1 = -\frac{d}{2}p^s + l/2.$$

Since $F^{(s-1)*}V_C$ is semistable, by Lemma 5.2, we have

$$\deg\ \mathcal{L}_1 - \deg\ \mathcal{M}_1 = l \leq 2g-2.$$

Since $0 \subset \mathcal{L}_1 \subset F^{s*}V_C$ is the strongly stable HN filtration, Theorem 4.12 and a calculation like that made in case (2) gives the desired value of $HKM(C)$. This proves the theorem. $\square$

If X is a nonsingular plane curve, then by Corollary 3.5, the bundle $V_{\mathcal{O}_X(1)}$ is semistable, and so Theorem 5.3 gives the following corollary.

Corollary 5.4. *Let X be a nonsingular plane curve of degree d over an algebraically closed field of characteristic $p > 0$, and $\mathcal{O}_X(1)$ the corresponding very ample line bundle. Then*

$$HKM(X, \mathcal{O}_X(1)) = \frac{3d}{4} + \frac{l^2}{4dp^{2s}},$$

where $s \geq 1$ is a number such that $F^{(s-1)*}V_{\mathcal{O}_X(1)}$ is semistable and $F^{s*}V_{\mathcal{O}_X(1)}$ is not semistable (if $F^{t*}V_{\mathcal{O}_X(1)}$ is semistable for all $t \geq 0$, we take $s = \infty$) and l is an integer congruent to pd (mod 2) with $0 \leq l \leq d(d-3)$.

Remark 5.5. If all the singularities of an irreducible projective plane curve of degree $d > 1$ are nodes and cusps, and the number of singularities is $\leq d-2$, then, by Corollary 3.6, it follows that Case (2) of Theorem 5.3 can not occur.

Remark 5.6. Suppose C is an irreducible projective plane curve with a singularity of multiplicity $= r \geq d/2$. Monsky conjectured

$$HKM(C) = \frac{3d}{4} + \frac{(2r-d)^2}{4d}.$$

We proved this in [T1]; note that it is an immediate consequence of cases (1) and (2) of Theorem 5.3, combined with Proposition 3.7.

Remark 5.7. Let C be an irreducible plane quartic. If C is singular, the last remark shows that $HKM(C)$ is 3 if C has a point of multiplicity 2, and is 13/4 if C has a triple point.

If C is nonsingular, then we are either in case (1) of Proposition 5.3, or in case (3) of the same proposition with $l = 2$ or 4. So $HKM(C)$ is either 3, $3 + (1/p^s)$ or $3 + (1/4p^{2s})$, for some $s \geq 1$. This result had been conjectured by Monsky.

In particular, when C is nonsingular, we have $HKM(C) \leq 3 + (1/p^2)$. The referee informs us that when $p = 2$, we have $HKM(C) \leq 3 + (1/16)$.

We recall some results of Monsky [M1], [M3] (see also [M2]), about nonsingular quartics of a certain type..

Theorem 5.8. *(Monsky) Let $R_\alpha = k[x, y, z]/(g_\alpha)$, where char $k = 2$ and*

$$g_\alpha = \alpha x^2y^2 + z^4 + xyz^2 + (x^3 + y^3)z,$$

with $\alpha \in k \setminus \{0\}$. Then

$$HKM(R_\alpha) = 3 + 4^{-m(\alpha)},$$

where, for $\lambda \in k$ such that $\alpha = \lambda^2 + \lambda$, we define $m(\alpha)$ as follows:

$$\begin{aligned} m(\alpha) &= \text{deg of } \lambda \text{ over } \mathbb{Z}/2\mathbb{Z} \text{ if } \alpha \text{ is algebraic over } \mathbb{Z}/2\mathbb{Z} \\ &= \infty \text{ if } \alpha \text{ is transcendental over } \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

Theorem 5.9. *(Monsky) Let $R_\lambda = k[x, y, z]/(f_\lambda)$, where char $k = 3$ and*

$$f_\lambda = z^4 - xy(x + y)(x + \lambda y),$$

with $\lambda \in k \setminus \{0, 1\}$. Then

$$HKM(R_\lambda) = 3 + \frac{1}{p^{2d(\lambda)}},$$

where $d = d(\lambda)$ is the degree of λ over $\mathbb{Z}/3\mathbb{Z}$ (and $d = \infty$ if λ is transcendental over $\mathbb{Z}/3\mathbb{Z}$).

Note that $X_\alpha = \text{Proj } R_\alpha \xrightarrow{\pi} \mathbf{P}^2$ is a nonsingular plane quartic of genus 3. We also note that, given any integer $n \geq 2$ there exists an $\alpha \in \bar{\mathbb{F}}_2$ such that $m(\alpha) = n$. Similarly given any $n \geq 1$ there exists $\lambda \in \bar{\mathbb{F}}_3$ such that $d(\lambda) = n$.

Applying Corollary 5.4 to Example 5.8, we see that $F^{(n-1)*}V_\alpha$ is semistable and $F^{n+1*}V_\alpha$ is not. (The referee has shown that $F^{n*}V_\alpha$ is semistable). Hence we get the following.

Proposition 5.10. (i) *Given any integer $n \geq 2$, there exists a non-singular quartic curve $X_\alpha \subseteq \mathbf{P}^2_{\bar{\mathbb{F}}_2}$, given by the equation*

$$\alpha x^2y^2 + z^4 + xyz^2 + (x^3 + y^3)z = 0$$

where $m(\alpha) = n$, such that the vector bundle

$$V_\alpha = \Omega^1_{\mathbf{P}^2} \mid_{X_\alpha}$$

is a semistable vector bundle on X_α of rank 2 and degree -4, and the iterated Frobenius pullback $F^{n*}V_\alpha$ is not semistable, while $F^{(n-1)*}V_\alpha$ is semistable.

(ii) *Given any integer $n \geq 1$, there exists a non-singular quartic curve $X_\lambda \subseteq \mathbf{P}^2_{\bar{\mathbb{F}}_3}$, given by the equation*

$$z^4 - xy(x + y)(x + \lambda y)$$

where $d(\lambda) = n$, such that the vector bundle

$$V_\lambda = \Omega^1_{\mathbf{P}^2} \mid_{X_\lambda}$$

is a semistable vector bundle on X_α of rank 2 and degree -4, and the iterated Frobenius pullback $F^{n*}V_\lambda$ is not semistable, while $F^{(n-1)*}V_\lambda$ is semistable.

Remark 5.11. Let R_λ be as in Theorem 5.9, but with $p > 3$. Monsky [M3] has given a practical algorithm involving the iteration of a rational function, for calculating $HKM(R_\lambda)$. Together with our results, this lets one calculate the smallest power of F^* that destabilizes V_λ.

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